Holographic Renormalized Entanglement Entropy

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1. Entanglement entropy

The entanglement entropy (EE) is a quantity that can be defined in any QFT whose Hilbert space $\mathcal{H}$ can be decomposed into two or more parts:

$$\mathcal{H} = \mathcal{H}_A \times \mathcal{H}_{\overline{A}}$$

Given such a decomposition we can define the EE of subsystem A in a state with density matrix ρ by constructing the reduced density matrix

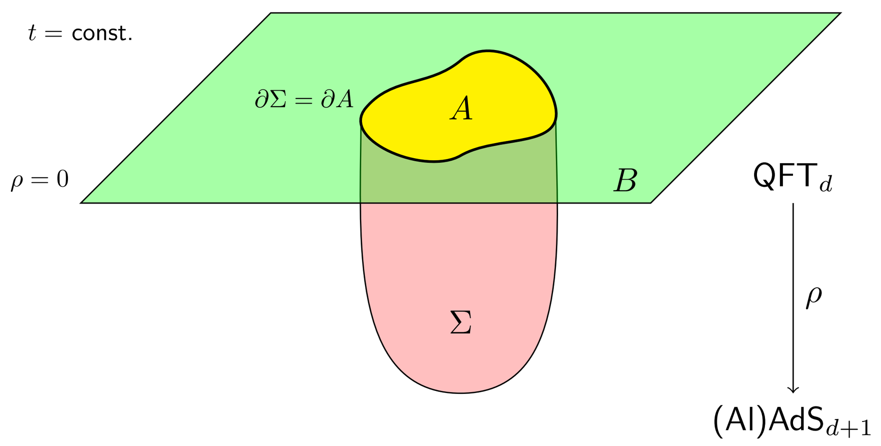
$$\rho_A = \text{tr}_{\mathcal{H}_{\overline{A}}} \rho$$

The EE is then defined as the von Neumann entropy of ρ_A

$$S_A = -\text{tr}\rho_A \log \rho_A$$

2. Holographic entanglement entropy

The Ryu-Takayanagi proposal claims we can calculate the EE of A holographically by finding the area of the minimal co-dimension 2 bulk surface Σ with boundary $\partial\Sigma = \partial A$ that is homologous to A:



The entanglement entropy is then given by

$$S_A = \frac{1}{4G_N} \int_\Sigma d^{d-1}\sigma \sqrt{\gamma}$$

3. UV divergences

The EE is UV divergent and needs a UV cutoff to be well defined:

$$S_A = \frac{c_{2-d}}{\varepsilon^{d-2}} \text{Area}(\partial A) + \cdots + \begin{cases} a \log\left(\frac{R}{\varepsilon}\right) + c_0 + o(\varepsilon^0) & d \text{ even} \\ c_0 + o(\varepsilon^0) & d \text{ odd} \end{cases}$$

where c_n and a are constants, R is some characteristic length scale, ε is the UV cutoff, and $\ldots$ denote subleading divergences. Notice the universal area law divergence at leading order. The coefficients a and c_0 are related to the a and F theorems in even and odd d respectively, they are scheme dependent however.

4. Previous renormalization attempts

The most popular renormalized EE is that of Liu & Mezei

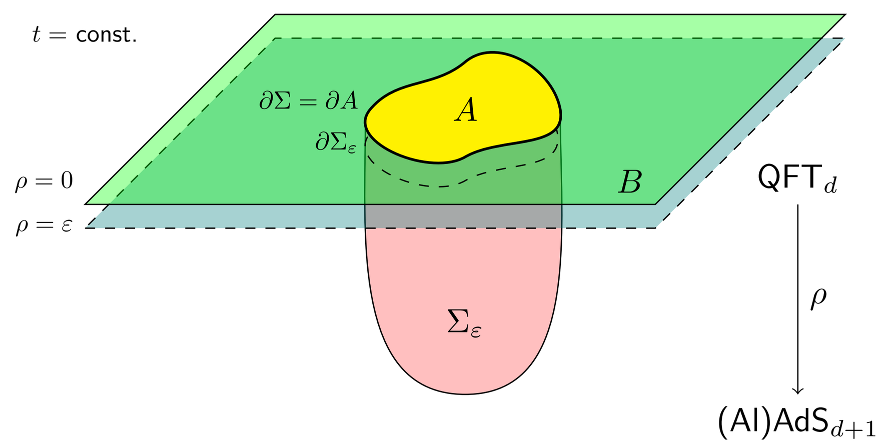
$$\mathcal{S}_A = \begin{cases} \frac{1}{(d-2)!!}\left(R\frac{d}{dR} - 1\right)\left(R\frac{d}{dR} - 3\right)\ldots\left(R\frac{d}{dR} - (d-2)\right) S_A & d \text{ odd} \\ \frac{1}{(d-2)!!} R\frac{d}{dR}\left(R\frac{d}{dR} - 2\right)\ldots\left(R\frac{d}{dR} - (d-2)\right) S_A & d \text{ even} \end{cases}$$

This definition is not perfect and has problems such as:

- It requires the geometry to have only one defining length scale, and does not generalise to more complex regions.
- The scheme dependence is obscure.
- $\mathcal{S}_A$ is not finite for non-CFTs, even for relevantly deformed CFTs.

5. Holographic renormalization

We can regularise the RT function by introducing a bulk cutoff surface



The renormalized entanglement entropy is then defined by

$$S_A = \lim_{\varepsilon \to 0} S_{A,\varepsilon} + S_{ct,\varepsilon}$$

where $S_{ct,\varepsilon}$ are covariant counter terms on $\partial\Sigma_\varepsilon$.

6. Counter terms in AdS_{D+2}

The first two terms in the counter term action for AdS_{D+2} are

$$S_{ct,\varepsilon} = -\frac{1}{4G_N}\frac{1}{D-1}\int_{\partial\Sigma_\varepsilon} \sqrt{\tilde{\gamma}}\left(1 + \frac{1}{2(D-1)(D-3)}\mathcal{K}^2\right)$$

Notice the first term is exactly what we expect from the area law divergence. More counter terms are needed in higher D, and logarithmic counter terms are needed in odd D.

7. Finite counter terms

We can find finite counter terms for all D, such as

$$\int_{\partial\Sigma_\varepsilon} d^{D-1}\sigma \sqrt{\tilde{\gamma}} \mathcal{K}^{D-1}$$

More terms are possible in higher dimensions, for example

$$\int_{\partial\Sigma_\varepsilon} d^{D-1}\sigma \sqrt{\tilde{\gamma}} \mathcal{R}^{(D-1)/2} \qquad \int_{\partial\Sigma_\varepsilon} d^{D-1}\sigma \sqrt{\tilde{\gamma}} (\mathcal{K}_{AB}\mathcal{K}^{AB})^{(D-1)/2}$$

are both finite for odd $D > 1$. Such terms account for scheme dependence in our results.

8. Relevant deformations

We can model RG flows by relevant deformations by adding some scalar fields $\phi_A(\rho)$ dual to a relevant scalar deformation. This introduces new divergences in the EE and so we must generalise the counter term action:

$$S_{ct,\varepsilon} = \int_{\partial\Sigma_\varepsilon} d^{D-1}\sigma \sqrt{\tilde{\gamma}} \mathcal{L}(\mathcal{R}, \mathcal{K}, \phi, \nabla\phi, \ldots)$$

Previous renormalization attempts failed to capture this dependence on matter fields.

9. Relevant deformations of AdS_4

For a single scalar deformation of dimension Δ, a logarithmic divergence appears at $\Delta = \frac{5}{2}$

$$S^{(\log)}_{ct,\varepsilon} = -\frac{1}{64G_N}\int_{\partial\Sigma_\varepsilon} \sqrt{\tilde{\gamma}}\phi^2 \log\varepsilon$$

to $O(\phi^2)$, and for $\Delta > \frac{5}{2}$ we need the counter term

$$S^{(\phi)}_{ct,\varepsilon} = -\frac{1}{4G_N}\int_{\partial\Sigma_\varepsilon} \sqrt{\tilde{\gamma}} \frac{3-\Delta}{8(5-2\Delta)}\phi^2$$

again to $O(\phi^2)$. No ϕ dependent counter terms are needed for $\Delta < \frac{5}{2}$.

10. References

S. Ryu and T. Takayanagi: [arXiv:hep-th/0603001]
H. Liu and M. Mezei: [arXiv:1202.2070 [hep-th]]
M. Taylor and WW: *to appear*

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