# Renormalized entanglement entropy

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# Outline

# What is entanglement entropy?

- Entanglement entropy is a measure of quantum entanglement between two complementary sub-systems.
- It can be defined for any quantum system that can be partitioned, in any state $\rho$.

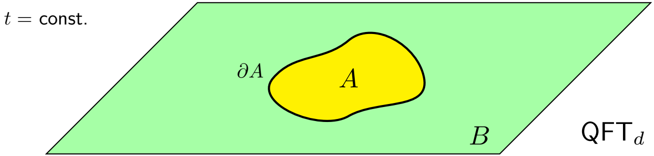
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

- Define the reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$.
- The entanglement entropy is the von Neumann entropy of $\rho_A$:

$$\boxed{S_{EE} = -\mathrm{Tr} \rho_A \log \rho_A}$$

# Holographic entanglement entropy

Entanglement entropy can be calculated holographically via the Ryu-Takayanagi prescription.



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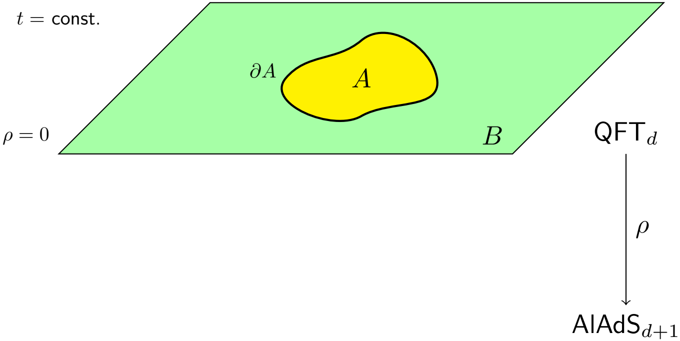

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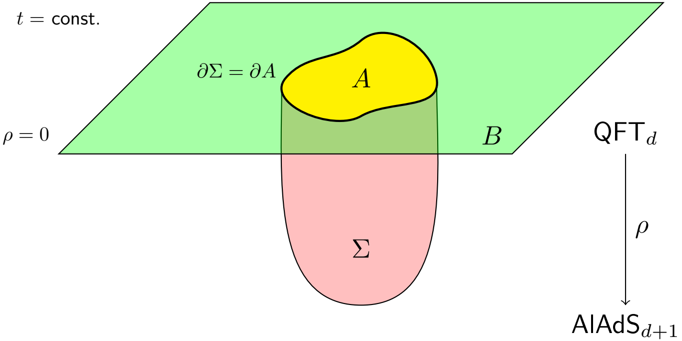

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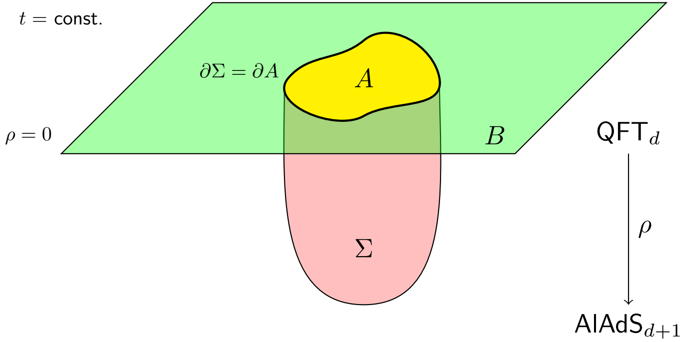


$$S_A = \frac{1}{4G}\int_\Sigma \mathrm{d}^{d-1}\sigma\sqrt{\gamma}, \qquad \gamma_{ab} = g_{\mu\nu}(X(\sigma))\partial_a X^\mu(\sigma)\partial_b X^\nu(\sigma)$$

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# Significance of entanglement entropy

- Does the EE capture global structure in the dual spacetime?
- What part of the bulk is reconstructable from a given boundary region?
- Related to the mutual information.
- Non-local order parameter for topological phase transitions.
- Possibly related to the $a$ quantity in odd $d$, and $F$ quantity in even $d$.

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- Possibly related to the $a$ quantity in odd $d$, and $F$ quantity in even $d$.
- The entanglement entropy is UV divergent!

# Divergences

- Area-law divergence ($d > 2$):

$$S_A = \frac{\gamma}{\epsilon^{d-1}} \text{Area}(\partial A) + \dots$$

- For a QFT in even dimensions $d = 1 + D$, the ground state EE contains universal terms

$$S_A \sim (-1)^{\frac{d}{2}-1} \, a \log\left(\frac{R}{\epsilon}\right)$$

where $R$ is a characteristic scale of $A$, $\epsilon$ is a UV cutoff, and $a$ is the $a$-theorem quantity.

- For odd $d$, finite terms

$$S_A \sim (-1)^{\frac{d-1}{2}} \, a$$

are conjectured to be related to the $F$ theorem, but are scheme dependent.

# Outline



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# Renormalization attempts

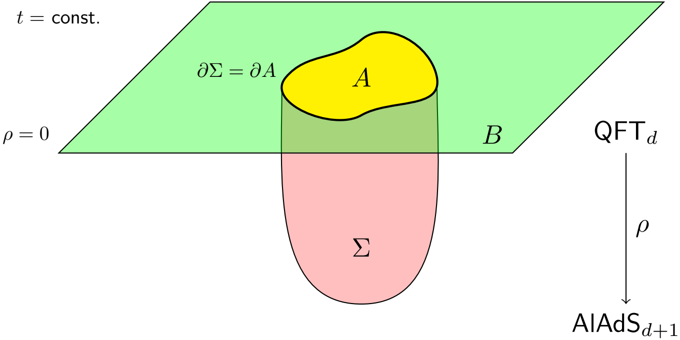
- Naïve subtraction (!)
- Differentiation with respect to parameters. (Cardy and Calabrese)
- Geometry dependence: e.g. in $d = 4$ with $\partial A$ sphere of radius $R$ then use

$$S_R = R\frac{\partial S}{\partial R} - 2S$$

  (Liu and Mezei)
- No definition for generic shape of $\partial A$.
- $S_R$ is not finite for non-CFTs, even relevantly deformed CFTs.
- Scheme dependence is obscure.

# Natural holographic renormalization scheme

There is a natural method of renormalizing quantities in holography.



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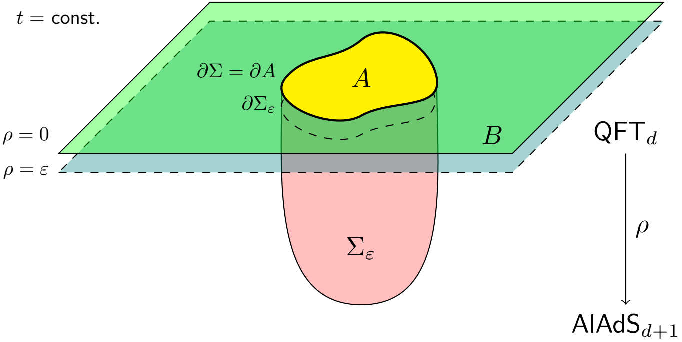


Use the cutoff $\rho = \varepsilon$ to define a renormalized volume using appropriate covariant counter terms.

# Holographic renormalization: Overview

1. Solve for the bulk fields.
2. Regulate the "bare" Ryu-Takayanagi functional:

$$S_{EE} = \frac{1}{4G_N} \int_{\Sigma} \mathrm{d}^{d-1}\sigma \sqrt{\gamma} \quad \longrightarrow \quad S_{EE,\varepsilon} = \frac{1}{4G_N} \int_{\Sigma_\varepsilon} \mathrm{d}^{d-1}\sigma \sqrt{\gamma}$$

   and expand $S_{EE,\varepsilon}$ as a power series in $\varepsilon$.

$$S_{EE,\varepsilon} = \frac{S_{(0)}}{\varepsilon^{d-1}} + \dots$$

3. Find covariant counter terms on $\partial\Sigma_\varepsilon$ to remove the divergences

$$S_{ct} \sim \int_{\partial\Sigma_\varepsilon} \mathrm{d}^{d-2}\sigma \sqrt{\tilde{\gamma}} \mathcal{L}(\mathcal{R}, \mathcal{K})$$

4. The renormalized entanglement entropy is then given by

$$S_{ren} = \lim_{\varepsilon \to 0} S_{EE,\varepsilon} + S_{ct}$$

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# Holographic renormalization: Step 1

For example, in an $AdS_{D+2}$ bulk:

1. Calculate the bulk fields near boundary expansions in Fefferman-Graham coordinates.

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad X^\mu(\sigma) = (\rho(\sigma), t, x^1(\sigma), \ldots, x^{D-1}(\sigma), y(\sigma))$$

2. Gauge fix the minimal surface coordinates $\sigma^a = (\rho, x^1, \ldots, x^{D-1})$.
3. Solve the minimal surface equation for $y(\rho, x)$ as a series expansion in $\rho$:

$$y(\rho, x) = y^{(0)}(x) + \rho\, y^{(1)}(x) + O(\rho^2)$$

$$y^{(1)}(x) = \frac{1}{2(D-1)} \left( y^{(0)}_{,AA} - \frac{y^{(0)}_{,A} y^{(0)}_{,AB} y^{(0)}_{,B}}{1 + {y^{(0)}_{,C}}^2} \right)$$

# Holographic renormalization: Step 2

Series expand $S_{EE,\varepsilon}$:

$$S_{EE,\varepsilon} = \frac{1}{4G_N}\int_{\partial\Sigma_\varepsilon} \mathrm{d}^{D-1}x\,\left(1 + {y^{(0)}_{,C}}^2\right)^{1/2} \times \left(\frac{\varepsilon^{-(D-1)/2}}{D-1} + \frac{\varepsilon^{-(D-3)/2}}{D-3}\frac{y^{(0)}_{,A}\,y^{(1)}_{,A} + 2{y^{(1)}}^2}{1 + {y^{(0)}_{,B}}^2} + \ldots\right)$$

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# Holographic renormalization: Step 3

Find scalars defined on $\partial\Sigma_\varepsilon$ that cancel divergences, starting with the highest order divegences.

- The volume form on $\partial\Sigma_\varepsilon$, $\sqrt{\tilde{\gamma}}$ is given by

$$\sqrt{\tilde{\gamma}} = \varepsilon^{-(D-1)/2}\left(1 + {y^{(0)}_{,C}}^2\right)^{1/2}\left(1 + \varepsilon\frac{y^{(0)}_{,A}y^{(0)}_{,A}}{1 + {y^{(0)}_{,C}}^2} + \ldots\right)$$

- This is sufficient for the first counter term:

$$S_{ct,1} = -\frac{1}{4G_N}\frac{1}{D-1}\int_{\partial\Sigma_\varepsilon} \mathrm{d}^{D-1}x\,\sqrt{\tilde{\gamma}}$$

- The trace of the extrinsic curvature is given by

$$\mathcal{K} = 2(D-1)\varepsilon^{1/2}\frac{y^{(1)}}{\left(1 + {y^{(0)}_{,C}}^2\right)^2} + \ldots$$

- This gives us the second counter term:

$$S_{ct,2} = -\frac{1}{4G_N}\frac{(D-2)}{2(D-1)^3(D-3)}\int_{\partial\Sigma_\varepsilon} \mathrm{d}^{D-1}x\sqrt{\tilde{\gamma}}\mathcal{K}^2$$

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# Holographic renormalization — Finite counter terms

- This method makes it very easy to find finite counter terms.
- The exists at least one such term for all $D$:

$$\int_{\partial\Sigma_\varepsilon} \mathrm{d}^{D-1}x\sqrt{\tilde{\gamma}}\mathcal{K}^{D-1}$$

- Higher dimensions allows a range of other finite counter temrs, all constructed from the curvature invariants, for example:

$$\int_{\partial\Sigma_\varepsilon} \mathrm{d}^{D-1}x\sqrt{\tilde{\gamma}}(\mathcal{K}_{AB}\mathcal{K}^{AB})^{(D-1)/2}$$

$$\int_{\partial\Sigma_\varepsilon} \mathrm{d}^{D-1}x\sqrt{\tilde{\gamma}}\mathcal{R}^{(D-1)/2}$$

are finite for odd $D$.

# Outline

# 3D CFT vacuum state

- The renormalised action for the entangling surface in $AdS_4$ is:

$$S_{ren} = \frac{1}{4G_N} \int_\Sigma \mathrm{d}^2\sigma \sqrt{\gamma} - \frac{1}{4G_N} \int_{\partial\Sigma} \mathrm{d}\sigma \sqrt{\tilde{\gamma}} \left(1 - c_s \mathcal{K}\right)$$

- Here $\mathcal{K}$ is the extrinsic curvature of the boundary curve (into the cut-off surface).
- This term is finite $\implies$ scheme dependence.

# Examples

- Half-plane

$$S_{EE} = 0$$

- Infinitely long strip of width $R$:

$$s_{EE} = -\frac{\pi^2\sqrt{2}}{3G_4 R}\frac{\Gamma(7/4)}{\Gamma(1/4)^2\Gamma(5/4)}$$

$\mathcal{K} = 0$ so no scheme dependence.

- Disc of radius $R$

$$S_{EE} = \frac{\pi}{2G_4}(a_s - 1)$$

$a_s$ measures scheme dependence.

# Extension to RG flows

- We can extend this framework to model RG flows.
- The bulk geometry is a domain wall

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + e^{2A(\rho)}\mathrm{d}x^i \mathrm{d}x_i$$

  with some scalar fields $\phi_A(\rho)$.
- Counter terms can and do depend on $\phi_A$:

$$S_{ct} \sim \int_{\partial\Sigma_\varepsilon} \mathrm{d}^{d-2}\sigma \sqrt{\tilde{\gamma}}\mathcal{L}(\mathcal{R}, \mathcal{K}, \phi_A, \nabla\phi_A, \ldots)$$

- Explains why previous attempts fail to handle relevant deformations.

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# 3D RG flows

Four dimensional bulk, $d = 3$, single scalar $\phi$ with square mass $m^2 = 3(3 - \Delta)$. We assume a relevant deformation: $\Delta < 3$.

- For $\Delta > 5/2$:

$$S_{ct} = -\frac{1}{4G_N}\int_{\partial\Sigma} \mathrm{d}\sigma \sqrt{\tilde{\gamma}} \left(1 - c_s \mathcal{K} + \frac{(3-\Delta)}{8(5-2\Delta)}\phi^2 + \dots\right)$$

- For $\Delta = 5/2$ this last term becomes anomalous:

$$S_{ct,\log} = -\frac{1}{128G_N}\int_{\partial\Sigma} \mathrm{d}\sigma \sqrt{\tilde{\gamma}}\phi^2 \log\varepsilon$$

- Conformal anomalies were found $\Delta = \frac{d}{2} + 1$ in general $d$, (e.g. Rosenhaus & Smolkin; Jones & Taylor).
- We find anomalies whenever $\Delta = \frac{6n-1}{2n}$ in $d = 3$.

# Change under relevant deformations

- Consider the change in the vacuum disc entanglment entropy due to a small relevant perturbation with dimension $\Delta$.
- The leading order change in the renormalized entanglement entropy is at $O(\phi_{(0)}^2)$ and given by

$$\delta S_{ren} = \frac{\pi}{16(2\Delta - 5)G_4}\phi_{(0)}^2 R^{2(3-\Delta)} + \dots$$

- This is positive if $\Delta > \frac{5}{2}$ and negative if $\Delta < \frac{5}{2}$.
- This suggests that the renromalized entanglement entropy is not a good $F$-quantity (work in progress. . . )

# Outline

# Summary and Outlook

- Proposed a renormalization of the holographic entanglement entropy using holographic renormalization.
  - Our method uses Fefferman-Graham expansions. Can we reformulate this in the dilatation expansion formalism?
- Scheme works for any type of boundary region, bulk manifold, spacetime dimensions. . .
- Showed that this is explicitly finite, and accounts for scheme dependence.
  - All known values seem to be negative. What does this mean?
  - Can we fix the scheme dependence on general grounds?
- Renormalized EE can depend explicitly on any matter fields.
  - Explains why previous attempts failed to handle non-CFT states.
  - Why is renormalization incompatible with the CHM map?